

Focal Elements Generated by the Dempster-Shafer Theoretic Conditionals: A Complete Characterization*

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Abstract – Incorporation of soft evidence into the fusion process poses considerable challenges, including issues related to the material implications of propositional logic statements, contradictory evidence, and non-identical scopes of sources providing soft evidence. The conditional approach to Dempster-Shafer (DS) theoretic evidence updating and fusion provides a promising avenue for overcoming these challenges. However, the computation of the Fagin-Halpern (FH) conditionals utilized in the conditional evidence updating strategies is non-trivial because of the lack of a method to identify the conditional focal elements directly. The work in this paper presents a complete characterization of the conditional focal elements via a necessary and sufficient condition that identifies the explicit structure of a proposition that will remain a focal element after conditioning. We illustrate the resulting computational advantage via several experiments.

Keywords: Dempster-Shafer theory, belief theory, conditional belief, evidence updating, evidence fusion.

1 Introduction

Incorporation of soft evidence (e.g., COMINT generated from communication chatter, HUMINT from informant and domain expert statements, OSINT from open-source intelligence) into the fusion process with an eye towards increased automation of the decision-making process is an issue that has recently attracted considerable attention from the evidence fusion community [1, 2]. The Dempster-Shafer (DS) belief theoretic framework [3] appears to possess certain advantages over other alternatives for capturing the types of imperfections that are more typical of soft evidence [4].

For example, DS theoretic models are able to preserve the material implication of propositional logic statements (viz., reflexivity, transitivity, and contra-positivity) that are represented by uncertain implication rules [5]. The DS theoretic conditional approach in [6] appears to be a suitable basis on which evidence updating and fusion strategies can be developed to overcome some of the challenges associated with

contradictory evidence, sources possessing large and possibly non-identical scopes, or frames of discernment (FoDs) to use the more familiar DS theoretic jargon [7–9].

The DS theoretic conditional notions utilized in the evidence updating strategies based on this conditional approach are the Fagin-Halpern (FH) conditionals [10]. The FH conditionals allow for a more appropriate probabilistic interpretation and a natural transition to Bayesian notions; the FH conditional belief and plausibility also correspond to the inner and outer measures of a non-measurable event [6, 7, 10].

However, the computation of the FH conditionals is non-trivial because there is no direct way to identify the conditional focal elements, i.e., propositions receiving non-zero support after conditioning. Thus one typically has to resort to computing the FH conditional beliefs of essentially all the propositions to identify the conditional focal elements. While the work in [6] provides several necessary conditions for a proposition to belong to the conditional core, i.e., the set of conditional focal elements, the lack of sufficient conditions render their effectiveness somewhat limited.

In this paper, for the first time, we provide a necessary and sufficient condition for a proposition to belong to the conditional core. This result establishes the explicit ‘structure’ of these conditional focal elements in terms of the conditioning event and the focal elements prior to conditioning. Thus, for a given core (i.e., the set of focal elements before conditioning) and a conditioning event, this result enables one to fully characterize the conditional core with no recourse to numerical computations.

This paper is organized as follows: Section 2 provides a review of essential DS theoretic notions; Section 3 contains our main result, the Conditional Core Theorem (CCT); Section 4 presents a discussion on the computational gains that one may obtain from this result; Section 5 contains several experimental simulations; and the concluding remarks appear in Section 6. An outline of the proof of the CCT is provided in Appendix A.

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2 Preliminaries

2.1 Basic Notions of DS Theory

Definition 1 Consider the FoD Θ and $A \subseteq \Theta$.

(i) The mapping $m : 2^\Theta \mapsto [0, 1]$ is a basic belief assignment (BBA) or mass assignment if $m(\emptyset) = 0$ and $\sum_{A \subseteq \Theta} m(A) = 1$. The BBA is said to be vacuous if the only proposition receiving a non-zero mass is Θ ; the BBA is said to be Dirichlet if the only propositions receiving non-zero mass are the singletons and Θ .

(ii) The belief of A is $Bl(A) = \sum_{B \subseteq A} m(B)$.

(iii) The plausibility of A is $Pl(A) = 1 - Bl(\bar{A})$. ■

While $m(A)$ measures the support assigned to proposition A only, $Bl(A)$ represents the total support that can move into A without any ambiguity; $Pl(A)$ represents the extent to which one finds A plausible. DS theory models the notion of *ignorance* by allowing the mass assigned to a composite proposition to move into its constituent singletons. The following notation is useful:

$$\mathfrak{F} = \{A \subseteq \Theta \mid m(A) > 0\}; \quad \hat{\mathfrak{F}} = \{A \subseteq \Theta \mid Bl(A) > 0\}. \quad (1)$$

In DS theoretic jargon, \mathfrak{F} is referred to as the *core*; $A \in \mathfrak{F}$ is referred to as a *focal element*. The triple $\mathcal{E} = \{\Theta, \mathfrak{F}, m(\cdot)\}$ is the corresponding *body of evidence (BoE)*. When focal elements are constituted of singletons only, the BBA, belief and plausibility all reduce to a probability assignment.

2.2 DS Theoretic Conditionals

Definition 2 (Fagin-Halpern (FH) Conditionals) [10]

For $\mathcal{E} = \{\Theta, \mathfrak{F}, m(\cdot)\}$, $A \in \hat{\mathfrak{F}}$, and $B \subseteq \Theta$, the conditional belief and conditional plausibility of B given A are

$$Bl(B|A) = Bl(A \cap B) \div [Bl(A \cap B) + Pl(A \cap \bar{B})];$$

$$Pl(B|A) = Pl(A \cap B) \div [Pl(A \cap B) + Bl(A \cap \bar{B})],$$

respectively. ■

Henceforth, when referring to the conditioning event $A \subseteq \Theta$, it is understood to belong to $\hat{\mathfrak{F}}$, i.e., $A \in \hat{\mathfrak{F}}$. Given the BoE $\mathcal{E} = \{\Theta, \mathfrak{F}, m(\cdot)\}$ and the conditional event $A \in \hat{\mathfrak{F}}$, we will denote the conditional core via $\mathfrak{F}|_A$:

$$\mathfrak{F}|_A = \{B \subseteq \Theta \mid m(B|A) > 0\}. \quad (2)$$

To get another useful expression for the FH conditionals, let $\mathcal{S}(A; B)$ denote the cumulative mass of propositions that ‘straddle’ between the two sets A and B , i.e.,

$$\mathcal{S}(A; B) = \sum_{\substack{\emptyset \neq X \subseteq A; \\ \emptyset \neq Y \subseteq B}} m(X \cup Y). \quad (3)$$

Note that $\mathcal{S}(A; C) \leq \mathcal{S}(A; B)$ whenever $C \subseteq B$. Using the relationship $Pl(A) - \mathcal{S}(\bar{A}; A \cap B) = Bl(A \cap B) + Pl(A \cap \bar{B})$, we may now express $Bl(B|A)$ as [6]

$$Bl(B|A) = Bl(A \cap B) \div [Pl(A) - \mathcal{S}(\bar{A}; A \cap B)]. \quad (4)$$

3 Conditional Core Theorem (CCT)

3.1 Preliminary Notions

We need the following notions to present our main result:

$$\text{out}(A) = \{B \subseteq A \mid B \cup C \in \mathfrak{F}, \emptyset \neq B, \emptyset \neq C \subseteq \bar{A}\};$$

$$\text{in}(A) = \{B \subseteq A \mid B \in \mathfrak{F}\}. \quad (5)$$

Using \mathcal{I} and \mathcal{J} to denote index sets that span the elements of $\text{out}(A)$ and $\text{in}(A)$, respectively, define the collections

$$\text{OUT}(A) = \left\{ B \subseteq A \mid B = \bigcup_{i \in \mathcal{I}} C_i, C_i \in \text{out}(A) \right\};$$

$$\text{IN}(A) = \left\{ B \subseteq A \mid B = \bigcup_{j \in \mathcal{J}} C_j, C_j \in \text{in}(A) \right\}. \quad (6)$$

So, to form an element of $\text{OUT}(A)$, from the focal elements that intersect but not contained in A , take the portions that intersect with A , and then take the union of some set of these portions; each element of $\text{IN}(A)$ is formed simply by the union of some set of focal elements each of which is contained in A . Note that, $B \subseteq A$, for all $B \in \text{IN}(A)$; moreover, $B \subseteq A$ and $\mathcal{S}(\bar{A}, B) > 0$, for all $B \in \text{OUT}(A)$.

As we will see, a set $B \subseteq A$ that can be expressed as $B = X \cup Y$, for some $X \in \text{in}(A)$, $Y \in \text{OUT}(A)$, is of special significance. For B to be expressed in this manner, we must of course have $X \subseteq B$ and $Y \subseteq B$.

We will also utilize the following notational convenience:

Definition 3 (Largest Set) Let \mathcal{S} be a set. For an arbitrary $s \in \mathcal{S}$ and some property $\mathcal{P} : \mathcal{S} \mapsto \{\text{true}, \text{false}\}$, we say that s^* is the largest element in \mathcal{S} satisfying the property \mathcal{P} and denote it as $s^* = LG(s \in \mathcal{S} \mid \mathcal{P})$, if $\mathcal{P}(s^*) = \text{true}$ and $\mathcal{P}(s') = \text{false}$, $\forall s' \in \mathcal{S}$ s.t. $s^* \subset s'$. By convention, we take $s^* = \emptyset$, if $\mathcal{P}(s) = \text{false}$, $\forall s \in \mathcal{S}$. ■

3.2 Main Result

We are now in a position to state our main result, the *Conditional Core Theorem (CCT)*, which establishes the necessary and sufficient conditions for a proposition to belong to the conditional core. The proof itself, in certain parts, is somewhat laborious and we relegate it to Appendix A.

Theorem 1 (Conditional Core Theorem (CCT)) Given the BoE $\mathcal{E} = \{\Theta, \mathfrak{F}, m(\cdot)\}$, $m(B|A) > 0$, $A \in \hat{\mathfrak{F}}$, iff B can be expressed as $B = X \cup Y$, for some $X \in \text{in}(A)$, $Y \in \text{OUT}(A) \cup \{\emptyset\}$. ■

The implications of the CCT can be summarized as follows:

- (a) If proposition B is not contained in the conditioning event, B cannot belong to the conditional core.
- (b) If proposition B is contained in the conditioning event,
 - (b.1) if B also belongs to the core, B belongs to the conditional core too.

(b.2) if B does not belong to the core, B belongs to the conditional core iff it can be expressed as the union of —a focal element contained in the conditioning event; and —the intersection of the conditioning event A with some arbitrary set of focal elements each of which straddles the conditioning event and its complement.

Some interesting direct consequences of the CCT are the following:

(a) If there are no focal elements that straddle the conditioning event and its complement, then the conditional core is equivalent to the core.

(b) Propositions with zero belief does not belong to the conditional core.

Example Consider the BoE, $\mathcal{E} = \{\Theta, \mathfrak{F}, m(\cdot)\}$ with $\Theta = \{a, b, c, d, e, f, g, h, i\}$, $\mathfrak{F} = \{a, b, h, df, beg, \Theta\}$ and $m(B) = \{0.1, 0.1, 0.1, 0.2, 0.2, 0.3\}$, for $B \in \mathfrak{F}$ (in the same order given in \mathfrak{F}). Then, for $A = (abcde)$,

$$\begin{aligned} \text{in}(A) &= \{a, b\}; & \text{out}(A) &= \{d, be, abcde\}; \\ \text{IN}(A) &= \{a, b, ab\}; & \text{OUT}(A) &= \{d, be, bde, abcde\}. \end{aligned}$$

Note that, $B = \{ad, bd, be, abe, bde, abde, abcde\}$, are the only propositions that can be expressed as $B = X \cup Y$, for some $X \in \text{in}(A)$ and $Y \in \text{OUT}(A)$. So, according to the CCT, $\mathfrak{F}|_A = B \cup \text{in}(A)$; or equivalently, the nine elements of \mathcal{B} and $\text{in}(A)$ are the only propositions that will belong to the conditional core (when $A = (abcde)$ is the conditioning proposition).

With the elements in the conditional core identified, we can use the expression for $\text{Bl}(B|A)$ in Definition 2 to get the conditional masses in Table 1. Without the benefit of the CCT, one would have to use this expression $2^{|A|} - 2 = 30$ times (instead of the $9 - 1 = 8$ times that we used). ■

Table 1: Conditional Core Corresponding to the Conditioning Proposition $A = (abcde)$

B	$m(B A)$	B	$m(B A)$	B	$m(B A)$
a	0.11110	b	0.11110	ad	0.03175
bd	0.03175	be	0.03175	abe	0.03175
bde	0.02540	$abde$	0.02540	$abcde$	0.60000

4 Computational Gains

Since propositions that are not contained in the conditioning event get zero conditional mass (see [6] and the CCT), to determine the conditional core $\mathfrak{F}|_A$ completely, it is only necessary to compute the conditional masses of $2^{|A|} - 2$ subsets of A only (note the two conditions $m(\emptyset|A) = 0$ and $\sum_{C \subseteq A} m(C|A) = 1$). The CCT however precisely identifies the conditional focal elements, thus eliminating the need for computing conditional masses of all these subsets of A . In fact, the exact number of computations to be made is

$$|\mathfrak{F}|_A| = |\{B = X \cup Y \text{ s.t. } X \in \text{in}(A), Y \in \text{OUT}(A) \cup \emptyset\}|. \quad (7)$$

4.1 Upper Bound on the Cardinality of $\mathfrak{F}|_A$

We can easily establish the following upper bound on the cardinality of the conditional core.

Claim 2 Let $N_{\text{in}} = |\text{in}(A)|$, $N_{\text{out}} = |\text{out}(A)|$, and $N_{\text{OUT}} = |\text{OUT}(A)|$. Then, $|\mathfrak{F}|_A| \leq N_{\text{in}}(1 + N_{\text{OUT}}) \leq N_{\text{in}}2^{N_{\text{out}}}$.

Proof: Let $\mathfrak{B} = \{B = X \cup Y \text{ s.t. } \emptyset \neq X \in \text{in}(A), \emptyset \neq Y \in \text{OUT}(A)\}$. Then,

$$\begin{aligned} |\mathfrak{F}|_A| &= |\text{in}(A) \cup \mathfrak{B}| = |\text{in}(A) \cup (\mathfrak{B} \setminus \text{in}(A))| \\ &= |\text{in}(A)| + |\mathfrak{B} \setminus \text{in}(A)| \\ &= |\text{in}(A)| + |\mathfrak{B}| - |\mathfrak{B} \cap \text{in}(A)| \\ &\leq N_{\text{in}} + N_{\text{in}}N_{\text{OUT}} - |\mathfrak{B} \cap \text{in}(A)| \\ &\leq N_{\text{in}} + N_{\text{in}}N_{\text{OUT}} \leq N_{\text{in}} + N_{\text{in}}(2^{N_{\text{out}}} - 1). \end{aligned}$$

This establishes the claim. ■

Remark: If $|\text{in}(A) \cap \mathfrak{B}| \ll N_{\text{in}}(1 + N_{\text{OUT}})$, then $|\mathfrak{F}|_A| \leq N_{\text{in}}(1 + N_{\text{OUT}})$ provides a tighter bound. Also, note that N_{OUT} could be significantly smaller than $2^{N_{\text{out}}} - 1$, since elements of $\text{OUT}(A)$ are not necessarily disjoint. Thus, $N_{\text{in}}2^{N_{\text{out}}}$ may lead to a very conservative bound if there is significant overlap in the elements of $\text{OUT}(A)$.

The example above shows an approximate reduction of 70% in required number of conditional mass computations. The CCT can be used to obtain significant computational savings in situations where the sets involved are of higher cardinality (e.g., when dealing with soft evidence where there is more uncertainty), and N_{in} and N_{out} are significantly small compared to $2^{|A|}$. For instance, take $|\Theta| = 50$, $|A| = 20$, $N_{\text{in}} = 500$, $N_{\text{out}} = 8$, and $N_{\text{OUT}} = 200$. The number of conditional mass computations reduces from $2^{20} - 2$ to a maximum of $500 \times 201 \leq 500 \times 2^8 = 128000$ (corresponding to about 88% reduction).

4.2 On Computational Time

While the CCT can be used to identify the conditional focal elements with no recourse to numerical computations, the identification step itself takes non-zero computational power. Indeed, with the CCT, the number of times the conditional mass has to be computed is typically much less; however, one also has to take into account the overhead associated with identifying conditional focal elements. To study this, let us define the following:

$\mathcal{T}_{\text{id.io}}$: average cpu time required to identify elements of $\text{in}(A)$ and $\text{out}(A)$.

$\mathcal{T}_{\text{id.cf}}$: average cpu time required to identify a proposition as a conditional focal element.

$\mathcal{T}_{\text{cp.cf}}$: average cpu time required to compute conditional mass of a proposition using the FH conditional expression.

Let

$$\alpha = \frac{\mathcal{T}_{\text{id.io}}}{\mathcal{T}_{\text{cp.cf}}}; \quad \beta = \frac{\mathcal{T}_{\text{id.cf}}}{\mathcal{T}_{\text{cp.cf}}}. \quad (8)$$

So, the computational time without using the CCT is

$$\mathcal{T}_{std} = (2^{|A|} - 1) \mathcal{T}_{cp.cf}. \quad (9)$$

The total time when using the CCT is

$$\mathcal{T}_{thm} = 2^{|\Theta|} \mathcal{T}_{id.io} + N_{in} 2^{N_{out}} \mathcal{T}_{id.cf} + \mathcal{T}_{cp.cf} |\mathfrak{F}|_{|A|}. \quad (10)$$

If $\mathcal{T}_{thm} \ll \mathcal{T}_{std}$, then the CCT can be used to reduce the computational burden. Once a BoE \mathcal{E} and a conditioning event A are given, the following procedure can be used to systematically check whether the application of the CCT is useful.

4.2.1 Criterion 1

Evaluate N_{in} and N_{out} . This only takes $2^{|\Theta|} \mathcal{T}_{id.io}$ of computational time. Then check if

$$\begin{aligned} N_{in} 2^{N_{out}} &< \frac{(2^{|A|} - 1) \mathcal{T}_{cp.cf} - 2^{|\Theta|} \mathcal{T}_{id.io}}{\mathcal{T}_{id.cf} + \mathcal{T}_{cp.cf}} \\ &= \frac{2^{|A|} - 1 - \alpha 2^{|\Theta|}}{1 + \beta}. \end{aligned} \quad (11)$$

If the above is true, then the CCT is guaranteed to provide computational gains; if it fails, check Criterion 2.

4.2.2 Criterion 2

Evaluate N_{OUT} . Then check if

$$\begin{aligned} N_{in} 2^{N_{out}} \mathcal{T}_{id.cf} + N_{in} (1 + N_{OUT}) \mathcal{T}_{cp.cf} \\ < (2^{|A|} - 1) \mathcal{T}_{cp.cf} - 2^{|\Theta|} \mathcal{T}_{id.io}, \end{aligned} \quad (12)$$

or equivalently, check if

$$\beta N_{in} 2^{N_{out}} + N_{in} (1 + N_{OUT}) < 2^{|A|} - 1 - \alpha 2^{|\Theta|}. \quad (13)$$

If the above is true, then the CCT is guaranteed to provide computational gains. If Criterion 2 fails, the wasted computational time is equal to $2^{|\Theta|} \mathcal{T}_{id.io} + N_{in} 2^{N_{out}} \mathcal{T}_{id.cf}$. This wastage is due to the identification time of conditional focal elements, which can be relatively large, especially with large FoDs and cores. If one wishes, the condition below can be checked before evaluating Criterion 2 to make sure that the wasted computational resources are negligible:

$$\frac{\alpha 2^{|\Theta|} + \beta N_{in} 2^{N_{out}}}{2^{|A|} - 1} \ll 1. \quad (14)$$

Note that, in an optimized implementation, we would have $\alpha \ll 1$ and $\beta \ll 1$.

5 Experiments

In this section, we carry out an experiment to analyze the performance of the CCT when it is used to compute the conditional masses. The results demonstrate how the cost associated with conditional mass computations is related to the cardinalities of the core and the conditioning event. Full knowledge of these relationships gives valuable information on how to optimize conditional mass computations for efficient implementation of evidence update and fusion strategies [9].

5.1 Setup

Fix the cardinality of the FoD as $N (= 7)$, i.e., $|\Theta| = N$. For a given $1 \leq n \leq 2^N$, randomly generate a BoE $\mathcal{E} = \{\Theta, \mathfrak{F}, m(\cdot)\}$ such that $|\mathfrak{F}| = n$. Then, for each conditioning event $A \subseteq \Theta$ s.t. $A \in \mathfrak{F}$, compute $m(B|A)$, $\forall B \subseteq A$. For each value of n , repeat this procedure $K (= 1000)$ times.

Let $\rho = |\mathfrak{F}| / (2^{|\Theta|} - 1)$. Also, we refer to computing all the $2^{|A|} - 1$ masses as the *standard method* of computing conditional masses.

5.2 Results and Analysis

Here, we study the effects of the number of focal elements and the cardinality of the conditioning event on the computational burden associated with computing the conditional masses.

5.2.1 Number of Computations Versus $|A|$

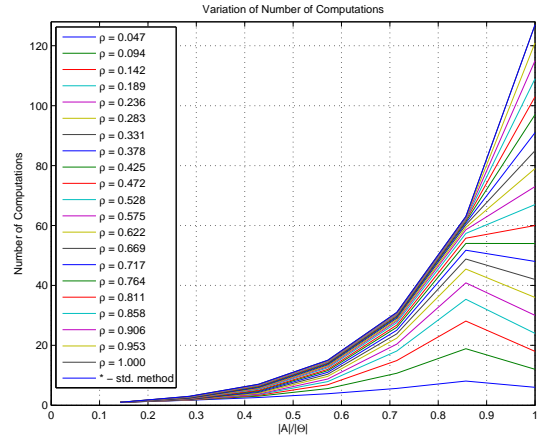


Figure 1: Number of computations versus $|A|/|\Theta|$ for different values of $\rho = |\mathfrak{F}| / (2^{|\Theta|} - 1)$.

Fig. 1 shows the number of computations versus $|A|$. Number of computations increases exponentially with $|A|$ in the standard method, irrespective of $|\mathfrak{F}|$. But, with the CCT, the number of computations reduce as the size of the core reduces. Note that the operational area will most likely be in the bottom right hand corner of Fig. 1, especially when dealing with soft data.

Fig. 2 gives a more detailed plot of the number of computations associated versus $|A|/|\Theta|$ for a fixed value of ρ . Clearly, use of the CCT provides a higher gain as $|A|$ increases.

5.2.2 Number of Computations Versus ρ

Fig. 3 shows the number of computations versus ρ .

Fig. 4, which is a more detailed plot corresponding to $|A| = 5$, clearly shows the advantage of the CCT. Gains are significant for smaller ρ , which is most often the case in real applications (e.g., for an FoD with $|\Theta| = 10$, a $|\mathfrak{F}| = 200$ would still yield $\rho < 0.2$).

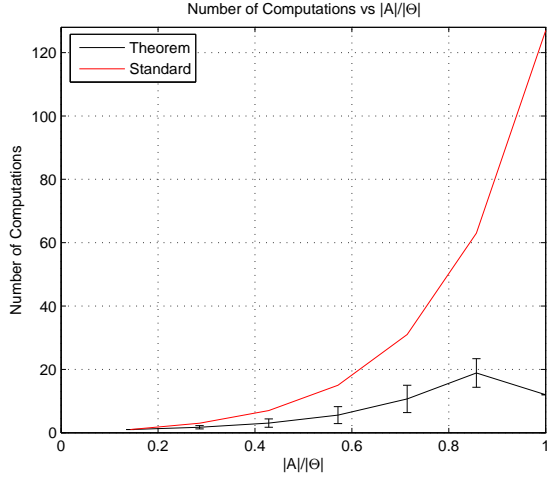


Figure 2: A detailed plot of the number of computations versus $|A|/|\Theta|$ when $|\mathcal{F}| = 12$ and hence $\rho = 0.0945 \approx 0.1$.

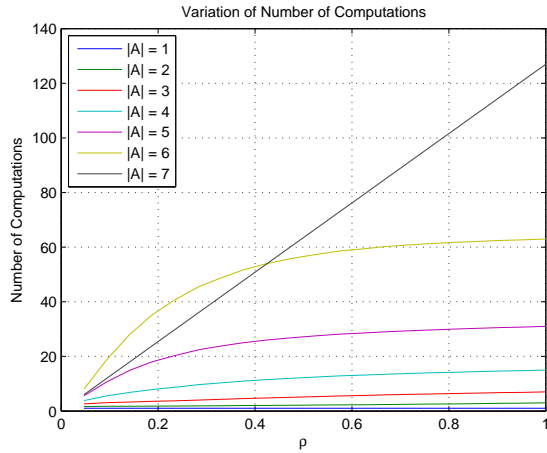


Figure 3: Number of computations versus ρ for different values of $|A|$.

5.2.3 Computational Time

Computational time associated with the CCT certainly depends on the implementation. In addition to the computation of the conditional masses themselves, the identification of conditional focal elements takes non-zero time. Computational time gains can be obtained when the total time of identification and mass computation is less than that of using the standard method. Our simulations were tested on a Apple Mac Pro Desktop with 2×2.66 GHz quad-core Intel Xeon processors, 16GB 1066 MHz DDR3 RAM, running Mac OS X 10.6.2. Programs are non-optimized and multiprocessing is not used. Further, computational times were obtained while running regular programs.

Fig. 5 shows the computational time versus $|A|$ when the CCT is used. The trend is similar to that of number of computations counterpart. In contrast to the standard method,

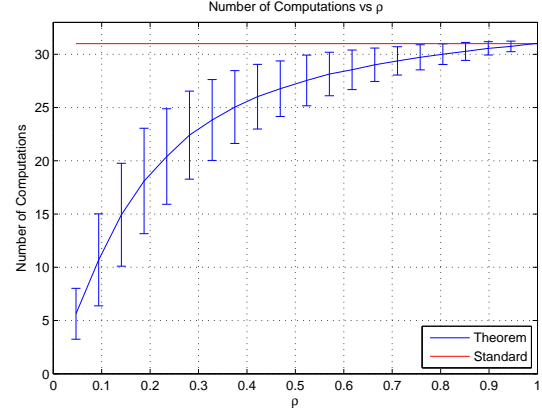


Figure 4: A detailed plot of the number of computations versus ρ when $|A| = 5$.

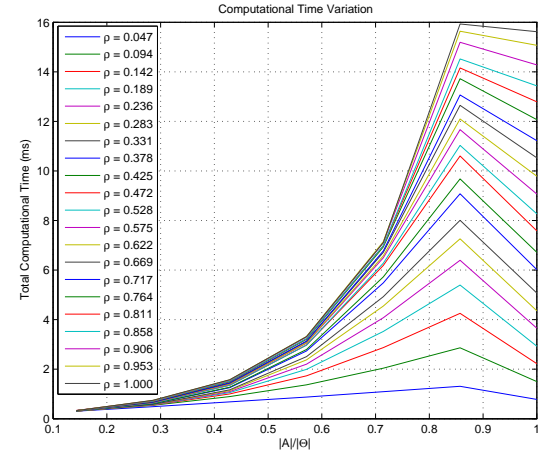


Figure 5: Computational time versus $|A|$ when the CCT is used. Multiple curves are shown for different values of $\rho = |\mathcal{F}|/(2^{|\Theta|} - 1)$.

computational time is dependent on ρ , and in turn on the number of focal elements. Here too computational time exponentially increases with respect to $|A|$. The decline towards the end (as $|A|$ reaches $|\Theta|$) is due to the decrease of N_{out} , which in turn results in a decrease of $|\mathcal{F}|_{|A|}$.

Fig. 6 is a more detailed plot for fixed value of $|\mathcal{F}|$. Use of the CCT clearly gives an advantage as $|A|$ increases. When $|A|$ is relatively smaller, as one would expect, the overheads associated with $\mathcal{T}_{id.io}$ and $\mathcal{T}_{id.cf}$ can be significant.

6 Concluding Remarks

The main result of this paper is the CCT which provides a complete characterization of the conditional focal elements. The CCT clearly reduces the number of computations required for conditional mass computations. But, the total computational cost, as we have pointed out, depends on the implementation. The total computational cost of identifying

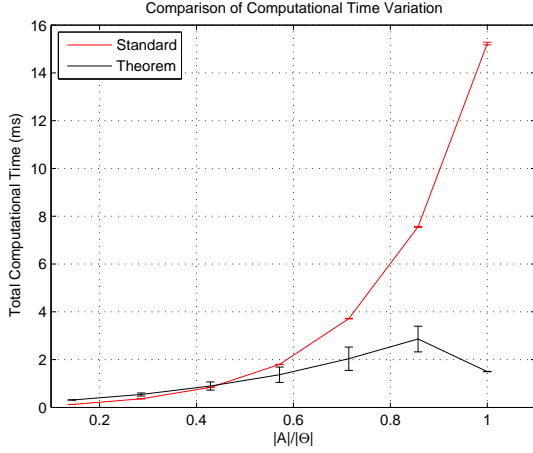


Figure 6: A detailed plot of the computational time versus $|A|/|\Theta|$ when $|\mathfrak{F}| = 12$ and hence $\rho = 0.0945$.

the elements of $\text{in}(A)$ and $\text{OUT}(A)$ is significantly smaller compared to the other two parameters and only depends on $|\Theta|$. But, as we observed, the total time required to identify the conditional focal elements can become significant under certain conditions. But we believe that it is possible to reduce this time by implementing an appropriate strategy to efficiently identify the focal elements. We intend to undertake this task in the future.

A Conditional Core Theorem: Proof

Due to the page limitations, we provide only an outline of the proof. We expect to provide a detailed proof in a future publication; upon request, it is also available from the authors.

Given $\mathcal{E} = \{\Theta, \mathfrak{F}, m(\cdot)\}$ and $A \in \hat{\mathfrak{F}}$, we have to prove the following:

$$m(B|A) > 0 \iff \exists X \in \text{in}(A), \exists Y \in \text{OUT}(A) \cup \{\emptyset\}, \\ \text{s.t. } B = X \cup Y.$$

Suppose $B \not\subseteq A$. Such a B cannot be expressed as $B = X \cup Y$, $X \in \text{in}(A)$, $Y \in \text{OUT}(A) \cup \{\emptyset\}$. This is consistent with $m(B|A) = 0$, $\forall B \not\subseteq A$ (see Lemma 8 of [6]).

Next, suppose $B \subseteq A$ and $B \in \mathfrak{F}$. Such a B can indeed be expressed as $B = X \cup Y$ by taking $X = B$, $Y = \emptyset$. This is consistent with $0 < m(B)/[\text{Pl}(A) - \mathcal{S}(\bar{A}; B)] \leq m(B|A)$ (see Lemma 10 of [6]).

So, from now on, we restrict our attention to the case when $B \subseteq A$ and $B \notin \mathfrak{F}$. With $B \notin \mathfrak{F}$, we must have $Y \neq \emptyset$, otherwise it would make $B = X$, so that $B \subseteq A$ and $B \in \mathfrak{F}$, a situation addressed above. Thus, we only have to prove the following:

$$m(B|A) > 0 \iff \exists X \in \text{in}(A), \exists Y \in \text{OUT}(A), \\ \text{s.t. } B = X \cup Y.$$

With no loss of generality, from now on, we can restrict our attention to $X \in \text{in}(A)$ s.t. $X \subseteq B$, and $Y \in \text{OUT}(A)$ s.t.

$Y \subseteq B$; it is impossible to represent B as $X \cup Y$ otherwise. Consider the ‘forward’ and ‘reverse’ directions separately.

A.1 ‘Forward’ Direction

Consider the contrapositive that corresponds to the forward direction:

$$(X \cup Y) \subset B, \forall X \in \text{in}(A), X \subset B, \\ \forall Y \in \text{OUT}(A), Y \subset B \implies m(B|A) = 0.$$

Neither $X = B$ nor $Y = B$ is considered because it implies that $X \cup Y = B$.

We proceed with the proof as follows. Pick $\hat{B} = \hat{X} \cup \hat{Y}$, where

$$\hat{X} = \bigcup_{\substack{X \in \text{in}(A); \\ X \subset B}} X; \hat{Y} = \text{LG} \{Y \in \text{OUT}(A) \mid Y \subset B\}.$$

Note that, $\hat{X} \subseteq B$, $\hat{Y} \subset B$, and $\mathcal{S}(\bar{A}; B) = \mathcal{S}(\bar{A}; \hat{Y})$. Also, $\hat{B} \subseteq B$. Let us consider the two cases, $\hat{B} \subset B$ and $\hat{B} = B$, separately.

A.1.1 $\hat{B} = (\hat{X} \cup \hat{Y}) \subset B$

Since $B \notin \mathfrak{F}$ and \hat{X} captures all the focal elements in B , $\text{Bl}(B) = \text{Bl}(\hat{X}) = \text{Bl}(\hat{B})$. This can be used to show that $\text{Bl}(B|A) = \text{Bl}(\hat{B}|A)$. Since $\hat{B} \subset B$, we conclude that $m(B|A) = 0$.

A.1.2 $\hat{B} = \hat{X} \cup \hat{Y} = B$

Pick $B_1 = X_1 \cup \hat{Y}$, where

$$X_1 = \text{LG} \{X \in \text{IN}(A) \mid X \cup \hat{Y} \subset B\}.$$

By construction, we can show the following:

$$X_1 \neq \emptyset; B_1 \subset B; \mathcal{S}(\bar{A}; B) = \mathcal{S}(\bar{A}; B_1); \\ m(C) = 0, \forall C = X \cup Y, \\ \text{s.t. } \emptyset \neq X \subset B \setminus B_1 \text{ and } Y \subseteq B_1.$$

Next pick $B_2 = X_2 \cup \hat{Y}$, where

$$X_2 = \text{LG} \{X \in \text{IN}(A) \mid (B \setminus B_1) \subseteq X, X \cup \hat{Y} \subset B\}.$$

Again, by construction, we can show the following:

$$X_2 \neq \emptyset; B \setminus B_1 \subset B_2 \subset B; \mathcal{S}(\bar{A}; B) = \mathcal{S}(\bar{A}; B_2); \\ m(C) = 0, \forall C = X \cup Y, \\ \text{s.t. } \emptyset \neq X \subset B \setminus (B_1 \cap B_2) \text{ and } Y \subseteq B_1 \cap B_2.$$

Further, by construction, we also notice the following:

$$(B \setminus B_1) \cup (B \setminus B_2) = B \setminus (B_1 \cap B_2); \\ B_1 \setminus (B_2 \setminus (B \setminus B_1)) = B \setminus B_2; \\ B_2 \setminus (B \setminus B_1) = B_1 \cap B_2.$$

We may repeat the above procedure until we have $X_{N+1} = \emptyset$, when

$$m(C) = 0, \forall C \text{ s.t. } \left(B \setminus \bigcap_{i=1}^N B_i \right) \subseteq C \subseteq B.$$

We can show that this procedure is guaranteed to terminate for some finite positive integer value N .

In the following discussion, we demonstrate the result for the $N = 2$ case; its generalization to arbitrary N is straight-forward. When $N = 2$, we have

$$m(C) = 0, \forall C, \text{ s.t. } B \setminus (B_1 \cap B_2) \subseteq C \subseteq B.$$

Then, we can show that

$$\text{Bl}(B) = \text{Bl}(B_1) + \text{Bl}(B_2) - \text{Bl}(B_1 \cap B_2),$$

which in turn can be used to show that

$$\text{Bl}(B|A) = \text{Bl}(B_1|A) + \text{Bl}(B_2|A) - \text{Bl}(B_1 \cap B_2|A). \quad (15)$$

On the other hand, working directly on conditional masses, we also obtain the following expression for $\text{Bl}(B|A)$:

$$\begin{aligned} \text{Bl}(B|A) &= \text{Bl}(B_1|A) + \text{Bl}(B_2|A) - \text{Bl}(B_1 \cap B_2|A) \\ &+ \sum_{\substack{\emptyset \neq C \subseteq B \setminus B_1; \\ \emptyset \neq D \subseteq B \setminus B_2}} m(C \cup D|A) \\ &+ \sum_{\substack{\emptyset \neq C \subseteq B \setminus B_1; \\ \emptyset \neq D \subseteq B \setminus B_2; \\ \emptyset \neq E \subseteq B_1 \cap B_2}} m(C \cup D \cup E|A). \end{aligned} \quad (16)$$

Compare (15) and (16):

$$\sum_{\substack{\emptyset \neq C \subseteq B \setminus B_1; \\ \emptyset \neq D \subseteq B \setminus B_2}} m(C \cup D|A) + \sum_{\substack{\emptyset \neq C \subseteq B \setminus B_1; \\ \emptyset \neq D \subseteq B \setminus B_2; \\ \emptyset \neq E \subseteq B_1 \cap B_2}} m(C \cup D \cup E|A) = 0.$$

Hence, we must have each individual term of both the summations equal to zero. In particular, when $C = B \setminus B_1$, $D = B \setminus B_2$, and $E = B_1 \cap B_2$, we have $C \cup D \cup E = B$, which yields $m(B|A) = 0$.

This establishes the ‘forward’ direction of the claim. \square

A.2 ‘Reverse’ Direction

Here, we need to prove the following:

$$\begin{aligned} \exists X \in \text{in}(A), X \subset B, \exists Y \in \text{OUT}(A), Y \subseteq B, \\ \text{s.t. } B = X \cup Y \implies m(B|A) > 0. \end{aligned}$$

Note that, we are ignoring the $X = B$ case because $B \notin \mathfrak{F}$.

Pick $G_1 = X_1 \cup Y_1$, where

$$\begin{aligned} X_1 &= \text{LG} \{X' \in \text{IN}(A) \mid X \subseteq X' \subset B\}; \\ Y_1 &= \text{LG} \{Y' \in \text{OUT}(A) \mid Y \subseteq Y', X_1 \cup Y' \subset B\}. \end{aligned}$$

By construction, we can show the following:

$$\begin{aligned} \mathcal{S}(\bar{A}; G_1) &< \mathcal{S}(\bar{A}; B); \\ G_1 &\subset B; \mathcal{S}(\bar{A}; B) > \mathcal{S}(\bar{A}; G_1); m(C|A) = 0, \\ \forall C &= D \cup E \text{ s.t. } \emptyset \neq D \subset B \setminus G_1, E \subseteq G_1. \end{aligned} \quad (17)$$

Next pick $G_2 = X_2 \cup Y_2$, where

$$\begin{aligned} X_2 &= \text{LG} \{X' \in \text{IN}(A) \mid B \setminus G_1 \subseteq X' \subset B\}; \\ Y_2 &= \text{LG} \{Y' \in \text{OUT}(A) \mid X_2 \cup Y' \subset B\}. \end{aligned}$$

Again, by construction, we can show the following:

$$\begin{aligned} \mathcal{S}(\bar{A}; B) &\geq \mathcal{S}(\bar{A}; G_2); \\ B \setminus G_1 &\subseteq G_2 \subset B; m(C|A) = 0, \\ \forall C &= D \cup E \text{ s.t. } \emptyset \neq D \subset B \setminus (G_1 \cap G_2), E \subseteq G_1 \cap G_2. \end{aligned} \quad (18)$$

We may repeat the above procedure until we have $X_{M+1} = \emptyset$, when

$$m(C|A) = 0, \forall C, \text{ s.t. } \left(B \setminus \bigcap_{i=1}^M G_i \right) \subseteq C \subset B.$$

We can show that this procedure is guaranteed to terminate for some finite positive integer value M .

As before, we show the result for the $M = 2$ case; its generalization to arbitrary M is straight-forward. When $M = 2$, we have

$$m(C|A) = 0, \forall C, \text{ s.t. } B \setminus (G_1 \cap G_2) \subseteq C \subset B.$$

The construction of sets G_i , $i = \overline{1, M}$, being similar to that of B_i , $i = \overline{1, M}$, as before, we can show that

$$\text{Bl}(B) = \text{Bl}(G_1) + \text{Bl}(G_2) - \text{Bl}(G_1 \cap G_2),$$

which in turn can be used to show that

$$\text{Bl}(B|A) > \text{Bl}(G_1|A) + \text{Bl}(G_2|A) - \text{Bl}(G_1 \cap G_2|A). \quad (19)$$

On the other hand, working directly on conditional masses, we also obtain

$$\begin{aligned} \text{Bl}(B|A) &= \text{Bl}(G_1|A) + \text{Bl}(G_2|A) \\ &- \text{Bl}(G_1 \cap G_2|A) + m(B|A). \end{aligned} \quad (20)$$

Compare (19) and (20) to conclude that $m(B|A) > 0$.

This establishes the ‘reverse’ direction of the claim. \square

This completes the proof of the CCT. \blacksquare

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